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FRACTURE WAVE IN A CHAIN STRUCTURE

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UDC 534.1:539.375

Various formulations of the problem of fracture wave propagation in an elastic brittle body are known (see [1-4] including references). Each of the proposed variants of the theory of this process is based on some hypothesis, for example: concerning the fracture wave velocity [2-5], the intensity of the elastic precursor [6], or the fracture energy [7, 8]. The introduction of an additional relation is necessary in order to close the system of equations of dynamics of the elasto-brittle continuum. However, such a relation cannot be justified without having recourse to data on the structure of the fracture front. This distinguishes the fracture wave from "ordinary" nonlinear waves whose macroparameters are determined independently of the structure of the front [8].

This fundamental difficulty can be overcome if we consider a structured medium, as is done below. As the simplest model of a structured medium we will take a linear chain in which each of the component unit masses interacts with the two adjacent masses through linear-elastic inertialess bonds (the distance between masses and the stiffnesses of the bonds are also taken as units of measurement, the velocity of the long waves in the undamaged chain being taken as the unit of velocity). At a certain bond stress $\sigma = \sigma_* \ll 1$ the bond partially fails: The bond stiffness takes a (positive) value $\alpha^2 < 1$. As distinct from the formulation of the same problem within a continuum framework [5-8], here there is no need to introduce any additional hypotheses.

Let us consider the stationary problem, in solving which we will use the same methods as in investigating the dynamics of a crack in a grid [9]. In the problem in question, taking into account the structure leads, essentially, to the same result as in [9]: high-frequency waves carrying part of the energy away from the fracture front (effect analogous to a temperature rise [7]). In the dynamics of a single crack, the structure of the medium determines the macroscopic fracture criterion and, consequently, affects the macroparameters of the velocity and stress fields. Thus, in the propagation of a fracture wave the microstructure also determines the macroparameters of the process (longwave approximation) — the ratios σ_1/σ_*, σ_2/σ_*, where σ_1 = const is the average stress in the elastic precursor, and σ_2 = const is the average stress behind the fracture front. The earlier assumptions [8] to the effect that $\sigma_1 < \sigma_*$ and fracture can occur even when $\sigma_2 < \sigma_*$ ($\sigma_{1,2} > 0$) are confirmed. The latter conclusion might appear strange if considered within the context of an unstructured continuum. Here, however, it is obvious: The total stresses behind the fracture front (with the high-frequency waves taken into account) exceed the average value.

1. Formulation of the Problem and Basic Relations

We assume that the velocities u and accelerations a are functions of a single variable $\tau = x - vt$, where $x = 0, \pm 1, \ldots$ are the Lagrangian coordinates of the masses, v = const > 0 is the velocity of the fracture front, and t is time. We note that the displacements, which owing to the presence of the elastic precursor also depend on x, cannot be similarly defined.

Leningrad, Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 128-134, November-December, 1984. Original article submitted September 24, 1982.

0021-8944/84/2506- 0921$08.50

In the case of undamaged bonds for an arbitrary mass we have the equation of motion $a(\tau) = Q(\tau) - Q(\tau - 1) + R(\tau)$, where $Q(\tau)$ is the increase in the distance between the given mass and the adjacent mass to the right (coordinate x + 1); $R(\tau)$ is the external force acting on the given mass. Then, $\partial^2 Q(\tau)/\partial t^2 = v^2 Q''(\tau) = a(\tau + 1) - a(\tau)$, where a prime denotes the derivative with respect to the argument. Hence we obtain the following equation "in the stresses" (for an undamaged grid the stress $\sigma(\tau) = Q(\tau)$):

$$v^2 Q''(\tau) + 2Q(\tau) - Q(\tau - 1) - Q(\tau + 1) = P(\tau) = R(\tau + 1) - R(\tau). \tag{1.1}$$

Let (partial) fracture of the bond occur at $\tau = 0$ ($t = x/v$), after which (at $\tau < 0$) $\sigma = \alpha^2 Q$. In order to take this into account, it is sufficient for $\tau < 0$ to compensate in (1.1) the stresses $(1 - \alpha^2)Q$ by means of the external forces, setting

$$P(\tau) = P_0(\tau) + (1 - \alpha^2)[2Q_-(\tau) - Q_-(\tau - 1) - Q_-(\tau + 1)],$$

$$Q_\pm(\tau) = Q(\tau)H(\pm\tau), \quad R(\tau) = R_0(\tau) + (1 - \alpha^2) \times [Q_-(\tau - 1) - Q_-(\tau)],$$

where H is the Heaviside unit function; $P_0(\tau) = R_0(\tau + 1) - R_0(\tau)$ is the external force stretching the bond in the problem with allowance for fracture (we will first consider the inhomogeneous problem and then turn to the homogeneous one). We assume that the external forces $P_0(\tau)$ are bounded. Then $Q(\tau)$ is continuous and the problem reduces to the equation

$$v^2 Q_+''(\tau) + 2Q_+(\tau) - Q_+(\tau - 1) - Q_+(\tau + 1) + v^2 Q_-''(\tau) + \alpha^2 (2Q_-(\tau) - Q_-(\tau - 1) - Q_-(\tau + 1)) = P_0(\tau) \tag{1.2}$$

with the condition

$$Q_+(0) = Q_-(0) = \sigma_* \quad (\sigma(+0) = \sigma_*, \quad \sigma(-0) = \alpha^2 \sigma_*). \tag{1.3}$$

In these relations it is possible to assume that the macrostress σ_2 behind the fracture front is given and determine σ_1, v and the oscillating waves behind the front and in the precursor. However, it is more convenient to assume that the velocity of the front v is given and determine the other corresponding quantities.

If the fracture of the bond occurs not in accordance with an external signal which could be received, say, at $\tau = 0$, but when the stress reaches a critical level $\sigma = \sigma_*$ ("natural fracture"), then the stationary solution sought must satisfy one further condition: The total stress at $\tau > 0$ (with allowance for the mean and oscillating waves) must be less than σ_*. We will assume that $\sigma_* > 0$ irrespective of whether the bond fractures in tension or compression.

2. General Solution

The solution of the stationary problem is nonunique. We will define it further by assuming that it is the limit as $t \to \infty$ (for an arbitrary finite neighborhood of $\tau = 0$) of the solution of the same, but nonstationary problem with zero initial conditions. The resulting solution selection rule was given in [10, 11]. Using this rule, after taking Fourier transforms

$$Q^F(q) = \int_{-\infty}^{\infty} Q(\tau) e^{iq\tau} d\tau$$

from Eq. (1.2) we obtain

$$h(iqv + 0, q) Q_+^F + \alpha^2 h(iqv\alpha^{-1} + 0, q) Q_-^F = P_0^F, \quad h(iqv + 0, q) = 2(1 - \cos q) + (0 + iqv)^2. \tag{2.1}$$

We will use the representation for h taken from [9, 10] (where the expression for h (2.1) was denoted as h^2, that for $h_\pm$ as $\mp h^2 i$) at $0 < v < 1$:

$$h(iqv + 0, q) = h_+(q, v) h_-(q, v),$$

$$h(q, v) = (0 \mp iq) \sqrt{1 - v^2}\, \Phi_\pm(iq, v) \prod_k L_k^\pm(v), \tag{2.2}$$

$$L_k^\pm(v) = 1 + \left[\frac{(0 \mp iq)}{\beta_k^\pm}\right]^2, \quad \beta_k^\pm = \beta_k^\pm(v), \quad \beta_k^\pm(v\alpha^{-1}) = \gamma_k^\pm(v).$$

Here the product includes the number of factors (greater by one for h_- than for h_+), which depends on v; $q = \beta_k^\pm$ are the positive roots of the equation $h = 0$, and $\sin \beta_k^+ \geqslant v^2 \beta_k^+$ (the roots correspond to oscillating waves whose group velocity v_g is not less than the phase velocity v), $\sin \beta_k^- \leqslant v^2 \beta_k^-$ ($v_g \leqslant v$); in the case of a double root (there are no roots of greater multiplicity) one of them is β_k^+, the other β_k^-; $\Phi_\pm(iq, v)$ are complex-conjugate functions of q that do not have zeros or poles at $\mathrm{Im}\, q > -\varepsilon < 0(\Phi_+)$ or at $\mathrm{Im}\, q < \varepsilon(\Phi_-)$.

Using the factorization (2.2), we represent (2.1) in the form ($0 < v < \alpha$)

$$\frac{h_+(q, v)}{h_+(q, v\alpha^{-1})} Q_+^F + \alpha^2 \frac{h_-(q, v\alpha^{-1})}{h_-(q, v)} Q_-^F = \frac{P_0^F}{h_-(q, v)\, h_+\left(q, \frac{v}{\alpha}\right)}. \tag{2.3}$$

We set

$$P_0^F = h_-(q, v)\, h_+(q, v\alpha^{-1}) \frac{2A\varepsilon}{q^2+\varepsilon^2}, \quad A = \text{const}\,(\varepsilon > 0). \tag{2.4}$$

From (2.1) and (2.2) it follows that in this case $P_0^F \to 0$ (and hence $P_0 \to 0$) as $\varepsilon \to 0$. Going over in the limit ($\varepsilon = +0$) to the homogeneous problem, we obtain

$$\frac{h_+(q, v)}{h_+(q, v\alpha^{-1})} Q_+^F + \alpha^2 \frac{h_-(q, v\alpha^{-1})}{h_-(q, v)} Q_-^F = A\pi\delta(q)\left(\frac{1}{0-iq} + \frac{1}{0+iq}\right), \tag{2.5}$$

where $\delta(g)$ is the Dirac function. In the class of bounded functions $Q_\pm(\tau)$ the solution of Eq. (2.5) is unique

$$Q_+^F(q) = \frac{A}{0-iq} \frac{h_+(q, v\alpha^{-1})}{h_+(q, v)}, \quad Q_-^F(q) = \frac{A}{\alpha^2(0+iq)} \frac{h_-(q, v)}{h_-(q, v\alpha^{-1})}. \tag{2.6}$$

Remark. There also exist other sequences of external forces, distinct from (2.4), that lead to certain homogeneous solutions:

$$P_0^F = h_-(q, v)\, h_+(q, v\alpha^{-1})\, K,$$

$$K = -\frac{4A\varepsilon}{(q^2+\varepsilon^2)^2}, \quad K = \frac{2A\varepsilon}{(q \pm \gamma_k)^2 + \varepsilon^2}, \quad K = \frac{2A\varepsilon}{(q \pm \beta_k)^2 + \varepsilon^2}, \quad (\varepsilon \to 0).$$

However, the first of these leads to a solution that does not satisfy the condition of boundedness of $Q_\pm(\tau)$, and the subsequent ones correspond to fracture under the action of the high-frequency waves (see [10]) when $\sigma_2 = \sigma_1 = 0$; this process is not considered here.

3. Longwave Approximation and Oscillating Waves

In view of the continuity of $Q(\tau)$ the elongation at fracture can be found, on the basis of (2.2) and (2.6), in the form

$$Q(0) = \sigma_* = \lim_{q \to +\infty} q \sqrt{Q_+^F(iq)\, Q_-^F(-iq)} = \frac{A}{\alpha\varkappa}, \quad \varkappa = \prod_k \frac{\beta_k^- \gamma_k^+}{\beta_k^+ \gamma_k^-}, \tag{3.1}$$

whence we can determine the constant A. The longwave approximation ($q \to 0$) has the form

$$Q_+^F \sim \frac{A}{0-iq} \sqrt{\frac{1-v^2\alpha^{-2}}{1-v^2}}, \quad Q_-^F \sim \frac{A}{\alpha^2(0+iq)} \sqrt{\frac{1-v^2}{1-v^2\alpha^{-2}}}. \tag{3.2}$$

Hence

$$\frac{\sigma_1}{\sigma_*} = \alpha\varkappa \sqrt{\frac{1-v^2\alpha^{-2}}{1-v^2}}, \quad \frac{\sigma_2}{\sigma_*} = \alpha\varkappa \sqrt{\frac{1-v^2}{1-v^2\alpha^{-2}}}. \tag{3.3}$$

The undamped oscillating waves propagating ahead of (behind) the fracture front are determined by the poles of expressions (2.6) for $Q_+(Q_-)$ on the real axis q at $q \neq 0$, i.e., by the zeros of the functions $h_+(q, v)(h_-(q, v/\alpha))$; $q = \pm\beta_k^+ (q = \pm\gamma_k^-)$. We find

$$Q_+(\tau) = \operatorname{Re} \frac{A h_+(\beta_k^+, v\alpha^{-1}) \exp(-i\beta_k^+\tau)}{i\beta_k^+ \sqrt{1-v^2}\, M_k^+(\beta_k^+)\, \Phi_+(i\beta_k^+, v)} =$$

$$= -A \sqrt{\frac{1-v^2\alpha^{-2}}{1-v^2}} \operatorname{Re}\left(\frac{N_+(\beta_k^+)}{M_k^+(\beta_k^+)} \exp(i\beta_k^+\tau)\right),$$

$$M_k^\pm(q) = M_\pm(q)\left[1 - \left(\frac{q}{\beta_k^\pm}\right)^2\right]^{-1}; \quad N_k^\pm(q) = N_\pm(q)\left[1 - \left(\frac{q}{\gamma_k^\pm}\right)^2\right]^{-1},$$

$$M_\pm(q) = \prod_m \left[1 - \left(\frac{q}{\beta_m^\pm}\right)^2\right], \quad N_\pm(q) = \prod_m \left[1 - \left(\frac{q}{\gamma_m^\pm}\right)^2\right].$$

The amplitude of the stress wave

$$|\sigma_{+k}| = A \sqrt{\frac{1-v^2\alpha^{-2}}{1-v^2}} \left| \frac{N_+(\beta_k^+)\, \Phi_+(i\beta_k^+, v\alpha^{-1})}{M_k^+(\beta_k^+)\, \Phi_+(i\beta_k^+, v)} \right|.$$

In accordance with (2.2) at $q \neq 0$

$$|\Phi_+(iq, v)| = \left(\frac{|h(iqv, q)|}{(1-v^2)q^2|M_+(q)M_-(q)|}\right)^{1/2},$$

$$|\Phi_+(iq, v\alpha^{-1})| = \left(\frac{|h(iqv\alpha^{-1}, q)|}{(1-v^2\alpha^{-2})q^2|N_+(q)N_-(q)|}\right)^{1/2}.$$

In the limit as $q \to \beta_k^+$

$$|\Phi_+(iq, v)| \to |\Phi_+(i\beta_k^+, v)| = \left|\frac{\sin\beta_k^+ - \beta_k^+ v^2}{(1-v^2)M_k^+(\beta_k^+)M_-(\beta_k^+)\beta_k^+}\right|^{1/2},$$

$$\Phi_+(iq, v\alpha^{-1})| \to |\Phi_+(i\beta_k^+, v\alpha^{-1})| = \left|\frac{2(1-\cos\beta_k^+) - (v\beta_k^+\alpha^{-1})^2}{(1-v^2\alpha^{-2})(\beta_k^+)^2 N_+(\beta_k^+)N_-(\beta_k^+)}\right|^{1/2}.$$

Hence

$$|\sigma_{+k}| = \frac{\alpha\sigma_*\varkappa}{\beta_k^+}\left|\frac{h(i\beta_k^+ v\alpha^{-1}, \beta_k^+)N_+(\beta_k^+)M_-(\beta_k^+)}{[\beta_k^+\sin\beta_k^+ - (v\beta_k^+)^2]N_-(\beta_k^+)M_k^+(\beta_k^+)}\right|^{1/2} \quad (\tau > 0). \tag{3.4}$$

Similarly we find

$$|\sigma_{-k}| = \frac{\alpha\sigma_*\varkappa}{\gamma_k^-}\left|\frac{h(i\gamma_k^- v, \gamma_k^-)N_+(\gamma_k^-)M_-(\gamma_k^-)}{[\gamma_k^-\sin\gamma_k^- - (\gamma_k^- v\alpha^{-1})^2]N_k^-(\gamma_k^-)M_+(\gamma_k^-)}\right|^{1/2}. \tag{3.5}$$

The velocities of the masses are determined as follows. We have

$$a(\tau) = \partial u/\partial t = -vu'(\tau),\ a^F(q) = iqvu^F(q).$$

Taking into account the fact that $a(\tau) = \sigma(\tau) - \sigma(\tau - 1)(\sigma_+(\tau) = Q_+(\tau),\ \sigma_-(\tau) = \alpha^2 Q_-(t))$ and $a^F(q) = (1 - e^{iq})\sigma^F(q)$, we obtain

$$u^F = -2\sigma^F(qv)^{-1}\exp(iq/2)\sin(q/2) + 2\pi B\delta(q) \sim -\sigma^F v^{-1} + 2\pi B\delta(q)(q \to 0),\ B = \text{const}. \tag{3.6}$$

The average value of the velocity (longwave approximation) is determined as the inverse transform of the asymptotic form in (3.6):

$$u_1 = u_+ = -\sigma_1 v^{-1} + B,\ u_2 = u_- = -\sigma_2 v^{-1} + B.$$

The constant B is determined from the obvious condition $u_1 = -\sigma_1$; $B = (v^{-1} - 1)\sigma_1$. Thus,

$$u_1 = -\sigma_1,\ u_2 = (v^{-1} - 1)\sigma_1 - \sigma_2 v^{-1}. \tag{3.7}$$

The velocities of the masses in the undamped oscillating waves are obtained as the contributions of the corresponding poles of transform (3.6), i.e., by replacing the parameter q (explicitly written out in (3.6)) by $\beta_k^+(\gamma_k^-)$:

$$|u_{+k}| = \frac{2}{v\beta_k^+}\left|\sigma_{+k}\sin\frac{\beta_k^+}{2}\right| \quad (\tau > 0),\quad |u_{-k}| = \frac{2}{v\gamma_k^-}\left|\sigma_-\ \sin\frac{\gamma_k^-}{2}\right| \quad (\tau < 0). \tag{3.8}$$

4. Energy Flux

The stationary solution selection rule used above leads to undamped oscillating waves whose group velocity is greater (less) than the phase velocity being present only ahead of (behind) the fracture front. Accordingly, the energy flux corresponding to these waves originates in the fracture front. Moreover, the energy contained in the constant-intensity elastic precursor (longwave approximation) flows away from the fracture front. The only possible source of energy for these waves, and of the fracture energy proper — the energy expended on the sudden loss of bond stiffness — is the wave of constant intensity behind the fracture front whose energy is created by the work done by the stresses σ_2. Thus, the following energy flux relation must be satisfied:

$$A_1(1-v) + A_2 v + \sum_k A_{+k}(v_{gk} - v) + \sum_k A_{-k}(v - v_{gk}) + T_0 v = -\sigma_2 u_2, \tag{4.1}$$

where $A_1 = (u_1^2 + \sigma_1^2)/2 = \sigma_1^2$ is the energy density in the constant-intensity elastic precursor; $A_2 = (u_2^2 + \sigma_2^2\alpha^{-2})/2$ is the energy density in the average wave behind the fracture front; $A_{+k}(A_{-k})$ is the average energy density of the oscillating wave (v is its phase velocity, v_{gk} the group velocity, and $-v\beta_k^+(-v\gamma_k^-)$ the frequency):

$$A_{+k} = \frac{1}{4}\left(|u_{+k}|^2 + |\sigma_{+k}|^2\right),\ A_{-k} = \frac{1}{4}\left(|u_{-k}|^2 + \frac{|\sigma_{-k}|^2}{\alpha^2}\right),$$

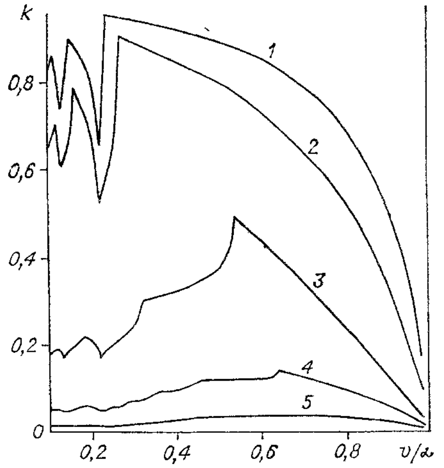

Fig. 1

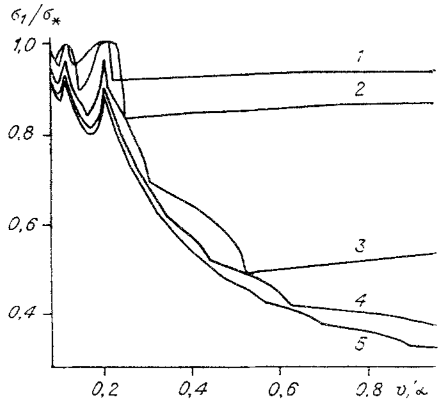

Fig. 2

$T_0 = (1 - \alpha^2)\sigma_*^2/2$ is the fracture energy.

Expressions for the amplitudes of the stresses and velocities concerned are given in (3.3)-(3.5), (3.7), and (3.8). Relation (4.1) can be (and was) used for checking the results of computations based on the above equations. Let us determine the ratio of the fracture energy T_0 to the total energy T dissipated at macrolevel in unit displacement of the fracture front. From (4.1), (3.7), and (3.3) there follows

$$T = T_0 + \sum_k (1 - v_{gk}v^{-1}) A_{-k} - \sum_k (1 - v_{gk}v^{-1}) A_{+k} =$$
$$= -\sigma_2 u_2 - A_1(1 - v) - A_2 v = \frac{1}{2v^2}(1 - v^2)\sigma_1(\sigma_2 - \sigma_1) =$$
$$= \frac{1}{2}(1 - \alpha^2)\sigma_*^2 \varkappa^2 = T_0\varkappa^2 = T_0 k^{-1}.$$

Thus

$$k = T_0/T = \varkappa^{-2}. \qquad (4.2)$$

The function $k(\alpha, v)$ completely determines the effect of the structure of the medium on the macroparameters of the fracture wave. Introducing this function is sufficient to close the equations of dynamics of an elasto-brittle unstructured continuum. We note that the expression for $k(v)$ applicable to a medium with a more general structure coincides with that given in [12].

5. Some Conclusions

Let $v_0 \leqslant v < 1$, $v_0 = 2|\sin(q_0/2)|/q_0 \approx 0.2106$ (q_0 is the minimum positive root of the equation $q_0 = 2\tan(q_0/2) \approx 8.987$). Then the equations $h(iqv_1 + 0, q) = 0$ ($v_1 = v$, $v_1 = v/\alpha \equiv w$) have in each case one positive root $q = \beta_1^- \equiv 2\beta(v_1 = v)$ and $q = \gamma_1^- \equiv 2\gamma(v_1 = w)$, $0 < \beta$, $\gamma < \pi$. The oscillating waves are propagated only behind the fracture front ($\tau < 0$). Turning to (3.1) and (3.3) and taking into account the equations satisfied by β and γ, we can write

$$\frac{\sigma_1}{\sigma_*} = \frac{\alpha\beta\sqrt{1 - v^2\alpha^{-2}}}{\gamma\sqrt{1 - v^2}} = \sqrt{f(\gamma) f^{-1}(\beta)}, \quad f(x) = \sin^{-2}x - x^{-2}.$$

On this interval $f(x)$ ($x = \beta, \gamma$) increases monotonically: $\partial f/\partial x > 0$, and $\gamma < \beta$ at $\alpha < 1$; therefore $\sigma_1/\sigma_* < 1(\alpha < 1)$.

Now, let $\alpha \to 0$, $v_0 < v/\alpha < 1$. Then the number of positive roots of the equation $h(iqv + 0, q) = 0$, equal to m for β_k^+ and m + 1 for β_k^-, increases without bound: $m \sim 1/\pi v \to \infty$, and the equation $h(iqv\alpha^{-1} + 0, q) = 0$ has only one positive root γ^-. In this case, with sufficient asymptotic accuracy it is possible to represent $\beta_k^\pm \sim 2\pi n \pm \arcsin(\pi n v)$, where n = 1, 2, ..., m for β_k^+ and n = 1, 2, ..., m + 1 for β_k^-, and find (see [11])

$$\varkappa^{-1} = \frac{\gamma_1^-}{\beta_{m+1}} \prod_{k=1}^{m} \frac{\beta_k^+}{\beta_k^-} \sim v\gamma_1^- \quad (v \to 0).$$

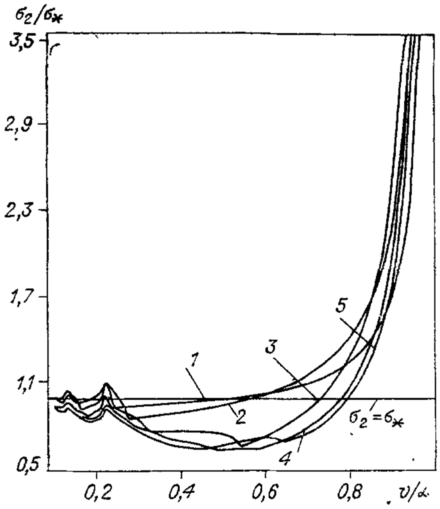

Fig. 3

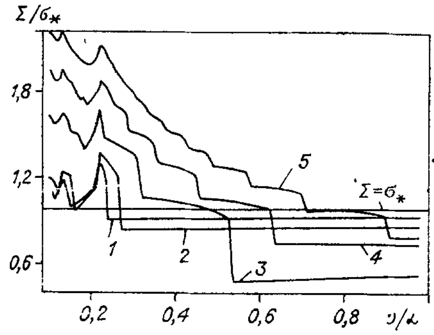

Fig. 4

In turn, for γ_1^- we have the asymptotic representation $\gamma_1^- \sim \sqrt{24(1 - v/\alpha)}\,(v \to \alpha)$, γ_1^- increasing monotonically with decrease in v and reaching at the lower limit of the interval in question the value $\gamma_1^- \approx 5.140$. Substituting these data in (3.3), we find $(\alpha \to 0)$

$$\frac{\sigma_1}{\sigma_*} \sim \frac{1}{\sqrt{12}} \approx 0.2887, \quad \frac{\sigma_2}{\sigma_*} \sim \frac{\alpha}{v}\,\frac{1}{4\sqrt{3}\,(1 - v\alpha^{-1})} \quad (v \to \alpha),$$
$$\frac{\sigma_1}{\sigma_*} \approx 0.9023, \quad \frac{\sigma_2}{\sigma_*} \approx 0.9442 \quad \left(\frac{v}{\alpha} = 0.2106\right)$$

(the factor α/v in the expression for σ_2/σ_* tends to unity, but has been retained in order to extend the range of v/α well described by this expression).

Thus, not only $\sigma_1 < \sigma_*$, but also under certain conditions $\sigma_2 < \sigma_*$.

In Fig. 1 we have plotted k as a function of v/α for $\alpha = 0.9, 0.8, 0.4, 0.2, 0.1$ (curves 1-5 respectively), and in Figs. 2 and 3 graphs of $\sigma_{1,2}(v/\alpha)$ for the same values of α.

Clearly, at sufficiently small values of σ_2 (but such that fracture can still occur, $\sigma_{2\min}(\alpha) < \sigma_2(v/\alpha) \leqslant \sigma_2(0)$) the velocity v is nonuniquely determined with respect to the value of σ_2.

It appears that the lower value of the velocity corresponds to instability of the branch $\sigma_2(v/\alpha)$. In this case, at any value of $\alpha < 1$ the velocity v has a nonzero lower bound (the existence of a lower bound for v was noted in [8]). For example, for $\alpha = 0.8$ and $\sigma_2 = 0.95$ the minimum $v = v_{\min} \approx 0.38$ (the velocity of the long waves in the undamaged grid has been taken as the unit of velocity).

In the event of natural fracture, apart from those found, at $\sigma_2/\sigma_* < 1$ there exists another static solution $v = 0$. The functions $k(v)$ are characterized by the same type of indeterminacy (the $k(v/\alpha)$ graphs are shown in Fig. 1).

It is also clear (see Fig. 2) that $\sigma_1/\sigma_* < 1$. The upper bound for the stresses ahead of the fracture front $\Sigma(v/\alpha)$ is equal to the arithmetic sum of σ_1 and the amplitudes of the oscillating waves with group velocities greater than v. In Fig. 4 we have plotted graphs of Σ/σ_* for the same values of α. Stationary solutions for the problem of natural fracture exist when this bound $\Sigma < \sigma_*$.

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STATISTICAL ESTIMATE OF BRITTLE STRENGTH WITH ALLOWANCE FOR CRACK RESISTANCE

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The classical statistical approach to the question of the probability of brittle fracture presupposes the presence in the investigated material of a system of defects, which also determines the strength of a specimen made of the material concerned. It is assumed that for each specific defect there is a corresponding local strength. The strength of the specimen as a whole (at least under static loading) is determined by the strength of the most dangerous defect, which in a given specimen has the minimum strength. The scale effect (SE) consists in the fact that in a specimen of greater volume there is a greater probability of encountering a more dangerous defect. Such an explanation of the SE was first given in [1], and a mathematical treatment using several different approaches was presented first in [2] and later in [3].

In [2] Weibull introduced the concept of the probability of fracture S_0 of unit volume at the stress σ, and on the basis of a solution of the statistical problem obtained the fracture probability S at stress σ for a specimen of volume V:

$$S = 1 - e^{-Vn(\sigma)}, \tag{1}$$

where the function $n(\sigma)$ is taken in the form

$$n(\sigma) = (\sigma/\sigma_0)^m \tag{2}$$

(σ_0 and m are experimentally selected material constants). Then, in [2] from Eq. (1), using (2), the following relation between the breaking stress and the volume of the test specimen was obtained:

$$\sigma_p = \sigma_0 I_m V^{-1/m},$$

where I_m is a constant for a given state of stress. In more general form

$$\sigma_p = A V^{-1/m}, \tag{3}$$

where $A = \sigma_0 I_m$.

Another approach to the solution of the problem is proposed in [3], namely to find the probability W(F)dF that in a specimen of volume V the most dangerous defect is that with the parameter $F \div F + \Delta F$

$$W(F)\,dF = \bar{n} V p(F) \left[\int_F^\infty p(F)\,dF\right]^{\bar{n}V-1} dF,$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 135-138, November-December, 1984. Original article submitted August 5, 1983.

0021-8944/84/2506-0927$08.50